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Dynamic Allocation Optimization of Palm Sap for Palm Sugar and Bioethanol Production Using Pontryagin Maximum Principle

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ABSTRACT

Palm sap supports renewable bioethanol production for Indonesia's energy needs while providing nutritious palm sugar with substantial export potential. Large-scale bioethanol production risks palm sugar sustainability, necessitating optimal dynamic allocation balancing energy demand and sugar production. This study models palm sap, sugar, and bioethanol dynamics, formulates optimal control, and analyzes existence, uniqueness, positivity, boundedness, and stability. The objective maximizes cumulative profit, using polynomial regression to forecast prices and profit margins to weight accumulated production. Optimal control uses Pontryagin's Maximum Principle and fourth-order Runge-Kutta forward-backward sweep computation. ver 24 months, bioethanol allocation exceeds palm sugar's, fluctuates with forecast prices, and converges to an equal 50–50% distribution. Dynamic allocation increases cumulative profit by 1.17% over static allocation.

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1. Introduction

Arenga Pinnata sap is a shared raw material for both palm sugar and bioethanol production, two commodities that serve distinct national needs: palm sugar helps offset Indonesia's projected sugar deficit of approximately 6.2 million tons per year while remaining a promising export commodity (Izzul, 2024; Mahrofi, 2025; Rahayu et al., 2025), whereas bioethanol supports the government's mandated 10%

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ethanol blend (E10) amid a substantial gap between national demand (696,000 kL/year) and current production capacity (40,000 kL/year) (Pinaria, 2025; Rahayu et al., 2025). Because both commodities draw from the same sap stock, the amount allocated to one at any given time directly constrains what remains available for the other, while the market prices of both commodities change independently over time.

Determining how much sap to allocate to each commodity at every point in time is therefore inherently a dynamic problem, an allocation decision made at one time shapes the sap stock and production outcomes available at every subsequent time. Such time-coupling and path dependency are defining features of dynamic optimization problems, which can be addressed through optimal control theory rather than static optimization.

Previous studies on palm sugar and bioethanol production from palm sap have focused on production processes, feasibility, and agroindustry potential (Aini et al., 2023; Evalia, 2004; Faliha et al., 2022; Hutami et al., 2023; Putri Intan Suri et al., 2024; Rahaju et al., 2023; Simamora et al., 2021) and bioethanol from palm sap (Ansar et al., 2021; Anwar et al., 2023; Rahayu et al., 2025; Riza et al., 2023), without formulating a mathematical model of the sap allocation process itself. Consequently, none of these studies offer a framework for determining how the allocation between the two commodities should evolve over time, nor for optimizing an economic objective subject to the system's underlying dynamics.

Because no comparable mathematical model of palm sap allocation currently exists, this study draws on the broader mathematical framework of dynamic resource allocation in plant systems (Kim & Yamauchi, 2026; Yegorov et al., 2017), in which the time-varying distribution of a shared resource among competing physiological demands is likewise modeled as a controlled dynamical system. Adapting this framework, the present study represents palm sap allocation between palm sugar and bioethanol production as a system of ordinary differential equations, coupled with Pontryagin's Maximum Principle to derive a time-varying allocation policy that maximizes cumulative economic profit, a modeling approach that has not previously been applied to the palm sap allocation problem specifically.

An optimal allocation strategy is then formulated using Pontryagin's maximum Principle (PMP), which transforms complex optimization problems in function spaces into pointwise optimization via the Hamiltonian function (Boscain et al., 2021), including over infinite time horizons relevant to long-term policy planning (Tauchnitz, 2015). PMP has been widely applied in solving optimal control problems in dynamical systems (Achsina & Mardlijah, 2021; Akbar & Mardlijah, 2024; Hanafi et al., 2024; Mardlijah et al., 2019; Mardlijah, Ariani, et al., 2017; Mardlijah, Jamil, et al., 2017; Ratianingsih et al., 2017; Santoso & Mardlijah, 2021; Sumardi & Hisyam, 2026) and dynamic resource allocation (Kim & Yamauchi, 2026; Yegorov et al., 2017). Thus, this study extends and integrates previous sector-specific investigations by providing a model-based decision-making framework for the optimal and sustainable management of palm sap allocation.

2. Method

The methodological stages of this study consist of problem identification and assumption formulation. Formulation of the mathematical model as a system of

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ordinary differential equations describing the dynamics of palm sap volume (N), palm sugar production (G), and bioethanol production (E), presented in detail in Section 3.1.

Analysis of the existence and uniqueness of the model's solution using the Banach Fixed Point Theorem; analysis of the positivity and boundedness of the model's solution; analysis of the model's stability using eigenvalue criteria; formulation of an optimal control objective function that maximizes profit by integrating palm sugar and bioethanol price prediction models obtained through polynomial regression; solution of the optimal control problem using Pontryagin's Maximum Principle; and numerical simulation using the forward-backward sweep method with a fourth-order Runge-Kutta scheme (FBSM-RK4). The mathematical model was constructed by modifying a dynamic plant resource allocation model (Kim & Yamauchi, 2026) given as follows,

$$\frac{dV}{dt} = axVUR$$
$$\frac{dU}{dt} = b(1-x)VUR$$
$$\frac{dR}{dt} = -cVUR$$

where R denotes the amount of resources or nutrients in the soil, V denotes aboveground plant biomass, U denotes belowground plant biomass, x denotes the rate of resource allocation to aboveground biomass and is the control variable, a denotes shoot growth efficiency, b denotes root growth efficiency, and c denotes the nutrient depletion rate.

This structure was adapted and modified into a bilinear system in the present study, under the assumption that the available palm sap volume is governed by a constant sap-harvesting rate, reduced by its rate of use for palm sugar and bioethanol production, each proportional to the control allocation proportion (u, v) and the available palm sap volume (N). The resulting quantities of palm sugar (G) and bioethanol (E) are governed by their conversion rate from palm sap according to the corresponding allocation proportion, reduced by the production failure rate. Unlike the reference model, which is trilinear (a product of three variables, V, U, R), the model in this study is bilinear, since the control variables (u, v) are multiplied directly by a single state variable (N).

The forward-backward sweep method with a fourth-order Runge-Kutta scheme (FBSM-RK4) was selected because it is a standard numerical method for solving the state and costate equation systems arising from the PMP conditions, and is capable of handling bilinear nonlinearity such as that present in this system, since the iteration alternates between the forward (state) and backward (costate) systems until the control converges. Convergence of the solution is ensured through the iterative criterion $\max |u^{(k)} - u^{(k-1)}| < \varepsilon$ with $\varepsilon = 10^{-5}$, following the convergence approach commonly used in this method (Lenhart & Workman, 2007).

The data used in this study were collected from secondary sources, including scientific articles, news articles, and the official websites of the Ministry of Energy and Mineral Resources, the National Research and Innovation Agency, and the Ministry of Agriculture. The research workflow is presented in the figure 1.

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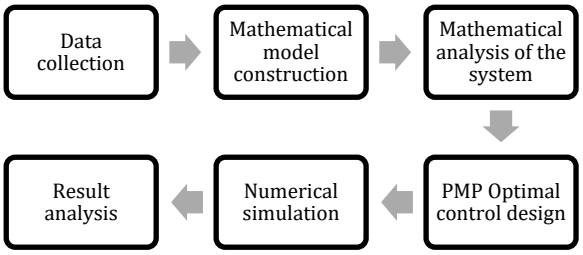


Figure 1. Research workflow

3. Results and Discussion

3.1. Mathematical Model

The mathematical model in this study is formulated based on the conversion of palm sap into palm sugar and bioethanol according to a specified allocation proportion. The allocation of palm sap between palm sugar and bioethanol production is governed by the control variables $u(t)$ and $v(t)$, representing the proportion of palm sap allocated to palm sugar and bioethanol production, respectively, subject to the constraint $u(t) + v(t) = 1$ and $0 < u(t), v(t) < 1$ for all t, since all available palm sap is assumed to be directly allocated and processed for the production of both commodities. This model consists of three time-dependent variables, namely $N(t)$ for the amount of palm sap at time t, $G(t)$ for the amount of palm sugar at time t, and $E(t)$ for the amount of bioethanol at time t. $N(t)$ increases by Λ due to the palm sap harvesting rate and decreases by $(\eta u + \theta v)N$ due to its allocation and processing for palm sugar and bioethanol production. $G(t)$ increases by $\alpha\eta uN$ as it is produced from palm sap and decreases by γG due to the palm sugar production failure rate. $E(t)$ increases by $\beta\theta vN$ as it is produced from palm sap and decreases by δE due to the bioethanol production failure rate. Grounded in the aforementioned assumptions, the mathematical model describing the dynamics of palm sap, palm sugar, and bioethanol quantities over time can be written as follows.

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - (\eta u + \theta v)N \\ \frac{dG}{dt} &= \alpha\eta uN - \gamma G \\ \frac{dE}{dt} &= \beta\theta vN - \delta E \end{aligned} \tag{1}$$

with $\Lambda, \eta, \theta, \alpha, \beta, \gamma, \delta, u, v > 0$ and initial values $N(t_0) = N_0, G(t_0) = G_0, E(t_0) = E_0$. The definitions of variables and parameters are presented in Tables of 1 and 2 as follows.

Table 1. Variable Definition

Variable	Definition	Unit of Measurement
N	Amount of palm sap	Liters
G	Amount of palm sugar	Kg
E	Amount of bioethanol	Liters
t	Time	Months

Table 2. Parameter Definition

Parameter	Definition	Unit of Measurement
Λ	Palm sap harvesting rate	liter month^{-1}

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Parameter	Definition	Unit of Measurement
η	Palm sap consumption rate for palm sugar production	month^{-1}
θ	Palm sap consumption rate for bioethanol production	month^{-1}
α	Conversion efficiency of palm sap into palm sugar	kg liter^{-1}
β	Conversion efficiency of palm sap into bioethanol	liter liter^{-1}
γ	Palm sugar production failure rate	month^{-1}
δ	Bioethanol production failure rate	month^{-1}
u	Proportion of palm sap allocation for palm sugar	
v	Proportion of palm sap allocation for bioethanol	

3.2. Mathematical Analysis

This section presents the analysis of the system and solution of the formulated model.

Theorem 3.1 (Existence and Uniqueness Solution). The system (1) with the given initial condition $\big(N(t_0), G(t_0), E(t_0)\big) \in \mathbb{R}^3$ has a unique solution $\big(N(t), G(t), E(t)\big)$ for all $t \in I = [t_0 - h, t_0 + h], h > 0$.

Proof. The proof procedure follows the steps outlined in the book by Kreyszig (1978) and handout written by Bryan (2011).

Define

$$\begin{aligned} f_1(N,G,E) &= \Lambda - (\eta u + \theta v)N \\ f_2(N,G,E) &= \alpha\eta u N - \gamma G \\ f_3(N,G,E) &= \beta\theta v N - \delta E \end{aligned} \tag{2}$$

Based on the fundamental theorem of calculus, and integrating both sides of the system of equation (2) from $s = t_0$ to $s = t < t_0 + h$ yields

$$N(t) = N_0 + \int_{t_0}^{t} f_1(N(s), G(s), E(s))ds$$

$$G(t) = G_0 + \int_{t_0}^{t} f_2(N(s), G(s), E(s))ds$$

$$E(t) = E_0 + \int_{t_0}^{t} f_3(N(s), G(s), E(s))ds$$

Let the vector $X(t) = \big(N(t), G(t), E(t)\big)$ and $f(X) = (f_1, f_2, f_3)^T$, so that the equations above can be written concisely as $X(t) = X_0 + \int_{t_0}^{t} f(X(s))ds$. Take any $\varepsilon > 0$, and define a compact set around the initial condition $K = [\max(0, N_0 - \varepsilon), N_0 + \varepsilon] \times [\max(0, G_0 - \varepsilon), G_0 + \varepsilon] \times [\max(0, E_0 - \varepsilon), E_0 + \varepsilon]$. The lower bound of each interval uses $\max(0,\cdot)$ to ensure that K contains no negative values, since N, G, E represent physical quantities that are non-negative nature. Since f_1, f_2, f_3 are polynomial functions, then $f \in C(\mathbb{R}^3, \mathbb{R}^3)$ with $f_i \in C(\mathbb{R}^3, \mathbb{R})$.

Define the Jacobian matrix $Df = \left[\frac{\partial f_i}{\partial X_j}\right]_{3\times 3}, i, j = 1,2,3$, with $X = (N, G, E)$. Since f_1, f_2, f_3 are polynomial, Df is continuous on K. By the Mean Value Theorem for

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vector-valued functions, if $\|Df(X)\|_\infty \leq L$ for all X in a compact convex set K, then f satisfies the Lipschitz condition on K with constant L, that is, $\|f(X_1) - f(X_2)\| \leq L(X_1 - X_2)$ for all $X_1, X_2 \in K$ (Khalil, 2002). Taking the infinity norm (maximum absolute row sum norm), we obtain $\|Df\|_\infty \leq L = \max\{R_1, R_2, R_3\}$, where $R_i = \sum_j \left|\frac{\partial f_i}{\partial X_j}\right|$ denotes the sum of absolute entries in row i of Df.

Hence, f satisfies the Lipschitz condition on K with the constant L defined above. Since f is continuous on the compact set K, f is bounded on K, that is, there exists a constant $M > 0$ such that $\|f(N, G, E)\| \leq M$ for all $(N, G, E) \in K$. Choose $h = \min(\frac{\varepsilon}{M}, \frac{1}{2L})$, define $I = [t_0 - h, t_0 + h]$, and let the continuous function space $C(I, \mathbb{R}^3)$ with the supremum norm $\|(N, G, E)\|_\infty = \sup_{t \in I}(|N(t)| + |G(t)| + |E(t)|)$ be a Banach space.

Define closed ball $Y = \big(N(t), G(t), E(t)\big) \in C(I, \mathbb{R}^3)$: $\|(N, G, E) - (N_0, G_0, E_0)\|_\infty \leq \varepsilon$ and the mapping

$$T(N, G, E)(t) = (N_0 + \int_{t_0}^{t} f_1\big(N(s), G(s), E(s)\big)ds\,, G_0 + \int_{t_0}^{t} f_2\big(N(s), G(s), E(s)\big)ds\,, E_0 + \int_{t_0}^{t} f_3\big(N(s), G(s), E(s)\big)ds)$$

for $\big(N(t), G(t), E(t)\big) \in Y$ and $t \in I$. Take any $(N_1, G_1, E_1), (N_2, G_2, E_2) \in Y$, since f satisfies the Lipschitz condition on K, it follows that

$$\|T(N_1, G_1, E_1) - T(N_2, G_2, E_2)\|_\infty \leq Lh\|(N_1, G_1, E_1) - (N_2, G_2, E_2)\|_\infty \leq \frac{1}{2}\|(N_1, G_1, E_1) - (N_2, G_2, E_2)\|_\infty$$

Therefore, T is contraction on Y. Since Y is a closed ball in a Banach space, then Y is a Banach space. Because T is a contraction on Y, the Banach fixed point theorem guarantees the existence of a unique fixed point $\big(N^*(t), G^*(t), E^*(t)\big) \in Y$ such that $T(N^*, G^*, E^*) = (N^*, G^*, E^*)$. Thus, it is proven that there exists a unique solution $\big(N^*(t), G^*(t), E^*(t)\big)$ satisfying system (1) on the interval $I = [t_0 - h, t_0 + h]$.

Theorem 3.2 (Positiveness of Solution). If the initial condition $(N(t_0), G(t_0), E(t_0))$ of system (1) belong to $\mathbb{R}^3_+$, then the corresponding solution remains in $\mathbb{R}^3_+$ for all $t \geq t_0$.

Proof. To establish the positivity of the solution, it is first demonstrated that $N(t) > 0$ for all $t \geq t_0$ by employing a proof by contradiction.

Suppose that there exists $t > t_0$ such that $N(t) \leq 0$. Since $N(t)$ is continuous function and $N(t_0) = N_0 > 0$, the set $A_N = \{t > t_0 | N(t) = 0\}$ is non-empty set. Define $t_1 = \inf A_N$, then $N(t) > 0 \ \forall t \in [t_0, t_1)$, $N(t_1) = 0$, and $\exists \epsilon > 0$ such that $N(t) \leq 0$ for $t \in (t_1, t_1 + \epsilon]$.

Let $P = \eta u + \theta v$. Since $\Lambda > 0$, we have

$$\frac{dN}{dt} = \Lambda - (\eta u + \theta v)N = \Lambda - PN > -PN \Leftrightarrow \frac{dN}{dt} + PN > 0$$

Multiplying both sides by the integrating factor e^{Pt}, and integrating over $[t_0, t_1]$, we obtain

$$e^{Pt_1}N(t_1) > e^{Pt_0}N(t_0)$$

Since $N(t_0) > 0$, $e^{Pt_0} > 0$, it follows that $e^{Pt_0}N(t_0) > 0$, hence $e^{Pt_1}N(t_1) > 0$. because $e^{Pt_1} > 0$, then $N(t_1) > 0$. However, this contradicts the condition that $N(t_1) = 0$. Therefore, the assumption is false, and it follows that $N(t) > 0 \ \forall t \geq t_0$. The proofs for $G(t)$ and $E(t)$ follow similar steps.

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Theorem 3.3 (Boundedness of Solution). Every solution of system (1) corresponding to positive initial conditions is bounded for all $t \geq 0$.
Proof. Consider system (1) with initial condition $N(0) = N_0, G(0) = G_0, E(0) = E_0$. The proof begins by showing that the solution of the first equation in system (1) is bounded for all $t \geq 0$. Given

$$\frac{dN}{dt} = \Lambda - (\eta u + \theta v)N$$

Multiplying both sides by the integrating factor $\mu(t) = e^{(\eta u+\theta v)t}$ and integrating both sides, we obtain

$$N(t) = \frac{\Lambda}{\eta u + \theta v} + \frac{C}{e^{(\eta u+\theta v)t}}$$

Substituting the initial condition yields $C = N_0 - \frac{\Lambda}{\eta u+\theta v}$, so the solution becomes

$$N(t) = \frac{\Lambda}{\eta u + \theta v} + (N_0 - \frac{\Lambda}{\eta u + \theta v})e^{-(\eta u+\theta v)t}$$

since $e^{-(\eta u+\theta v)t} \leq 1$ for $t \geq 0$, it follows that

$$|N(t)| \leq \max\{|N_0|, \frac{\Lambda}{\eta u + \theta v}\}$$

Thus, $\exists M_N = \max\{|N_0|, \frac{\Lambda}{\eta u+\theta v}\}$ such that $|N(t)| \leq M_N \ \forall t \geq 0$. Therefore, $N(t)$ is bounded.

The steps for proving the boundedness of $G(t)$ and $E(t)$ follow similarly, using the boundedness of $N(t)$. The results are as follows.
$\exists M_G = |G_0| + \frac{\alpha\eta u M_N}{\gamma}$ such that $|G(t)| \leq M_G \ \forall t \geq 0$, and $\exists M_E = |E_0| + \frac{\beta\theta v M_N}{\delta}$ such that $|E(t)| \leq M_E \ \forall t \geq 0$. Hence, $G(t)$ and $E(t)$ are bounded. Therefore, it is proven that the solution of system (1) is bounded.

Theorem 3.4 (Equilibrium and Stability). The system (1) has an equilibrium point (N^*, G^*, E^*) that is asymptotically stable.
Proof. The equilibrium point of system (1) is obtained by solving system (2) set equal to zero, yielding $(N^*, G^*, E^*) = (\frac{\Lambda}{\eta u+\theta v}, \frac{\alpha\eta u\left(\frac{\Lambda}{\eta u+\theta v}\right)}{\gamma}, \frac{\beta\theta v(\frac{\Lambda}{\eta u+\theta v})}{\delta})$. Thus, system (1) has an equilibrium point. For the purpose of this local stability analysis, the allocation proportion u and v are assumed constant over time (a fixed allocation policy), as distinct from the time-varying optimal control $u^*(t), v^*(t)$ derived in section 3.3 via Pontryagin's Maximum Principle. Under this assumption, system (1) reduces to a linear time-invariant system, so that the equilibrium point (N^*, G^*, E^*) is well defined and its stability can be analyzed using the standard eigenvalue criterion.

The stability of the equilibrium point is analyzed using the eigenvalue stability criterion at the origin. Since the equilibrium point of system (1) is not at the origin, it is shifted to the origin by defining $(n, g, e)^T = (N, G, E)^T - (N^*, G^*, E^*)^T$, so that the resulting error system, which is equivalent to system (1), satisfies $\left(\frac{dn}{dt}, \frac{dg}{dt}, \frac{de}{dt}\right)^T = \left(\frac{dN}{dt}, \frac{dG}{dt}, \frac{dE}{dt}\right)^T$. This leads to the following homogeneous system

$$\begin{bmatrix} \frac{dn}{dt} \\ \frac{dg}{dt} \\ \frac{de}{dt} \end{bmatrix} = \begin{bmatrix} -(\eta u + \theta v) & 0 & 0 \\ \alpha\eta u & -\gamma & 0 \\ \beta\theta v & 0 & -\delta \end{bmatrix} \begin{bmatrix} n \\ g \\ e \end{bmatrix} \quad (3)$$

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where u, v in the coefficient matrix above take cpnstant values under the fixed allocation policy assumption stated above, rather than the time-varying control variables $u(t), v(t)$.

The stability of the origin of system (3) is analyzed by examining the eigenvalues of the characteristic equation $\det(\lambda I - A) = 0$, where A is the coefficient matrix of system (3). The eigenvalues are $\lambda = -(\eta u + \theta v)$, $\lambda = -\gamma$, $\lambda = -\delta$. Since $\eta, u, \theta, v, \gamma, \delta > 0$, all eigenvalues are negative. Therefore, based on the stability theorem as presented by Subiono (2023), the origin of system (3) is asymptotically stable. This implies that the solution of system (3) approaches (0,0,0) as $t \to \infty$, which is equivalent to the solution of system (1) approaching the equilibrium point (N^*, G^*, E^*) as $t \to \infty$. Hence, the equilibrium point (N^*, G^*, E^*) is asymptotically stable.

3.3. Optimal Control

The objective of the optimal control problem in this study is to optimize the dynamic allocation proportion of palm sap for the production of palm sugar and bioethanol by maximizing the profit obtained from both commodities. Mathematically, this optimal control problem is formulated as maximizing the following objective function.

$$J(u, v) = \int_{t_0}^{t_f} \left(w_1 G + (w_2 + w_3) E - \frac{1}{2} w_4 u^2 N - \frac{1}{2} w_5 v^2 N \right) dt \quad (4)$$

where u and v are the control variable, w_1 (IDR/kg) denotes the profit from palm sugar production, w_2 (IDR/liter) denotes the profit from bioethanol production, w_3 (IDR/liter) denotes the environmental benefit of bioethanol utilization, obtained from the product of the social cost of carbon and the amount of carbon emissions avoided through bioethanol use, so that w_3 is expressed in the same monetary unit as w_2 and can be consistently summed with it, while w_4 and w_5 (IDR/liter) denote the palm sap allocation cost, multiplied by N since this cost arises from the actual volume of palm sap allocated to each commodity.

The profits from palm sugar and bioethanol production are obtained from the difference between price and production cost. Palm sugar prices (in IDR/kg) and bioethanol prices (in IDR/liter) are forecasted using polynomial regression based on historical monthly price data in Indonesia over the 2019–2024 period, processed using Minitab software. Polynomial regression was selected because the price trends of both commodities exhibit nonlinear patterns over time that cannot be adequately captured by simple linear regression.

For each commodity, first-, second-, and third-order polynomial models were compared and evaluated using Mean Absolute Percentage Error (MAPE), R^2, Adjusted R^2, and the t-test p-value for each coefficient. For the palm sugar price, the second-order model was selected as the best model (MAPE = 3.95%, $R^2 = 0.8810$, Adjusted $R^2 = 0.8775$, all coefficient significant at $p < 0.05$), since the third-order term in the alternative model was not significant ($p = 0.115$). For the bioethanol price, the third-order model was selected as the best model (MAPE = 6.14%, $R^2 = 0.6329$, Adjusted $R^2 = 0.6167$, all coefficients significant at $p < 0.05$). Both models were used to forecast prices starting from the 73rd data point, yielding the following palm sugar and bioethanol price functions.

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$$P_G(t) = 3.013(t+72)^2 - 95.7(t+72) + 17999$$
$$P_E(t) = 0.0786(t+72)^3 - 8.97(t+72)^2 + 329.6(t+72) + 8717$$

The optimal control problem is solved using the Pontryagin Maximum Principle by constructing the Hamiltonian function and maximizing it through the solution of the state equations, costate equations, and boundary conditions (Mardlijah, Jamil, et al., 2017). Based on equation (4) and the system of equation (1), the Hamiltonian function is obtained as follows.

$$H(N,G,E,u,v) = w_1 G + (w_2+w_3)E - \frac{1}{2}w_4 u^2 N - \frac{1}{2}w_5 v^2 N + \lambda_1(\Lambda - (\eta u + \theta v)N) + \lambda_2(\alpha\eta u N - \gamma G) + \lambda_3(\beta\theta v N - \delta E) \quad (5)$$

Thus, the state equations are obtained as follows,

$$\frac{\partial H}{\partial \lambda_1} = \Lambda - (\eta u + \theta v)N$$
$$\frac{\partial H}{\partial \lambda_2} = \alpha\eta u N - \gamma G$$
$$\frac{\partial H}{\lambda_3} = \beta\theta v N - \delta E$$

and the costate equations are given as follows.

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N} = \frac{1}{2}w_4 u^2 + \frac{1}{2}w_5 v^2 + \lambda_1(\eta u + \theta v) - \lambda_2\alpha\eta u - \lambda_3\beta\theta v$$
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial G} = -w_1 + \lambda_2\gamma$$
$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial E} = -(w_2+w_3) + \lambda_3\delta$$

This study assumes that palm sap is allocated only to palm sugar and bioethanol. Therefore, the optimal control problem is subject to the constraint $u + v = 1$. By substituting $v = 1 - u$ into the Hamiltonian function (5), the control equation is obtained as follows.

$$\frac{\partial H}{\partial u} = 0 \Leftrightarrow -w_4 u N + w_5(1-u)N - \lambda_1(\eta - \theta)N + \lambda_2\alpha\eta N - \lambda_3\beta\theta N = 0$$

$$\Leftrightarrow u = \frac{\lambda_1(\eta-\theta) - \lambda_2\alpha\eta + \lambda_3\beta\theta - w_5}{-(w_4+w_5)}$$

Since $u + v = 1$, $0 < u < 1$, $0 < v < 1$, then

$$f(x) = \begin{cases} 0, & \tilde{u}(t) \le 0 \\ \tilde{u}(t), & 0 < \tilde{u}(t) < 1 \\ 1, & \tilde{u}(t) \ge 1 \end{cases}$$

Thus, the optimal control $u(t)$ can be expressed as $u(t)^* = \min(1, \max(0, \tilde{u}(t)))$ or equivalently, $u^*(t) = \min(1, \max(0, \frac{\lambda_1(\eta-\theta) - \lambda_2\alpha\eta + \lambda_3\beta\theta - w_5}{-(w_4+w_5)}))$. The optimal control $v(t)$ is given by $v^*(t) = 1 - u^*(t)$. The optimal system is obtained by substituting $u^*(t)$ into the state dan costate equations.

3.4. Numerical Simulation

This section presents the numerical simulation outcomes of the optimal system along with the corresponding optimal control results. The numerical simulation is carried out using the initial values $N_0 = 170291000$ liters, $G_0 = 65496538$ kg, $E_0 = 51087300$ liters, the production cost of palm sugar is IDR 16,377/kg (Item et al.,

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2025), the production cost of bioethanol is IDR 8,250/liter (Martins, 2022), the environmental benefit of bioethanol utilization is IDR 8,967/liter (Cauley, 2022; Rahayu et al., 2025; Setiawan, 2022), and the allocation cost of palm sap is IDR 2,000/liter. The values of the parameters utilized in the simulation are provided in Table 3 as follows.

Table 3. Parameter Value

No.	Parameter	Value
1.	Λ	5108730000 liters month^{-1}
2.	η	30 month^{-1}
3.	θ	30 month^{-1}
4.	α	0.1 kg liter^{-1}
5.	β	0.06 liter liter^{-1}
6.	γ	3.9 month^{-1}
7.	δ	3 month^{-1}
8	w_1	IDR $P_G(t) - 16377$ kg^{-1}
9	w_2	IDR $P_E(t) - 8250$ liter^{-1}
10	w_3	IDR 8967 liter^{-1}
11	w_4	IDR 2000 liter^{-1}
12	w_5	IDR 2000 liter^{-1}

The parameter values used in the numerical simulation were determined based on the following data and references. The palm sap harvesting rate (Λ) was calculated from a sap productivity of 11 liters/tree/day and a total arenga palm plantation area of 61,924 hectares in Indonesia (Humas BRIN, 2024), with an agroforestry planting pattern of 4×10 meters yielding 250 trees per hectare (Helmy, 2025), resulting in $\Lambda = 170291000$ liters/day or 5108730000 liters/month. The palm sap conversion rates (η, θ) were assumed to be 30/month, based on the assumption that all allocated palm sap is fully consumed (100%) in the daily production process. The palm sugar formation efficiency ($\alpha = 0.1$ kg/liter) was obtained from a conversion ratio of 10 liters of palm sap yielding 1 kg of palm sugar, and the bioethanol formation efficiency ($\beta = 0.06$ liter/liter) from a ratio of 15 liters of palm sap yielding 1 liter of fuel grade bioethanol (Humas BRIN, 2024; Susilo et al., 2019).

The palm sugar production failure rate ($\gamma = 3.9$/month) was obtained from the ratio of the production risk threshold to average production based on a palm sugar production risk analysis study (Widyantara, 2019). The bioethanol production failure rate ($\delta = 3$/month) was obtained from a worst-case estimate combining yield reduction due to bacterial contamination of 1–5% (Brexó & Sant'Ana, 2017) and suboptimal distillation temperature (Maidangkay & Dosoputranto, 2021). The palm sap allocation cost ($w_4 = w_5 = 2000$ IDR/liter) was estimated as 20% of the farm-gate palm sap price of IDR 10,000/liter (Long et al., 2022), based on the consideration that the production site is located near the palm plantation.

Figure 2 shows the dynamics of palm sap allocation for palm sugar and bioethanol production over 24 months. The allocation of palm sap for bioethanol is consistently higher than for palm sugar throughout the simulation period, consistent with the priority of meeting renewable energy needs without sacrificing palm sugar production. The palm sugar allocation proportion (u) increases from 9.4% at month 0 to a peak of approximately 29.4% around months 14-15, gradually

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declines to about 23.1% at month 22, then rises sharply to 33.3% at month 23 and reaches 50% at month 24 (Table 4). This non-monotonic pattern is a consequence of the PMP optimality condition, in which $u(t)^*$ at each time is determined by the interaction between the costate variables $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ that representing the marginal values of palm sap, palm sugar, and bioethanol stocks, and the profit margins $w_1(t)$ and $w_2(t) + w_3$, which vary over time following the price forecasts $P_G(t)$ and $P_E(t)$.

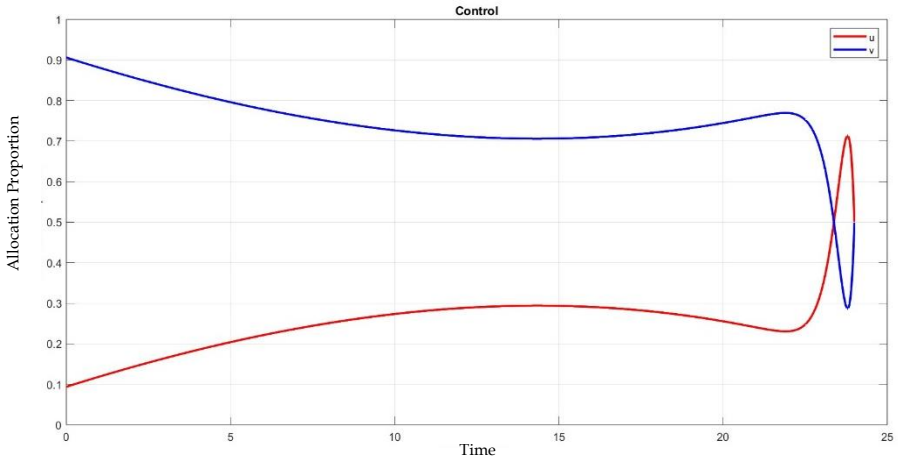


Figure 2. Dynamics of palm sap allocation for palm sugar and bioethanol production

Since the costates are solved backward from the terminal condition and depend simultaneously on all model parameters $(\eta, \theta, \alpha, \beta, \gamma, \delta)$, the resulting allocation pattern is non-monotonic, reflecting continuous adjustment to the price dynamics of both commodities over the simulation horizon, rather than a numerical artifact. The change from $u(23) = 0.333$ to $u(24) = 0.5$ is a direct consequence of the transversality condition for the finite-horizon optimal control problem with a free terminal state, namely $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = 0$. As $t \to t_f$, formula $u(t)^*$ reduce to $u(t_f)^* = \frac{w_5}{w_4 + w_5} = \frac{2000}{4000} = 0.5$, exactly matching the value at month 24. At the end of the horizon, there is no further incentive to anticipate profit beyond the simulation's scope, so the optimal control simply minimizes the quadratic allocation cost, which is symmetric since $w_4 = w_5$. This phenomenon is known as the end of horizon effect, a standard characteristic of finite-horizon optimal control problems (Lenhart & Workman, 2007), rather than an indication of numerical instability.

The allocation proportions for each t are presented in Table 4. Based on this optimal allocation strategy $J(u^*, v^*) =$ IDR 47,737,291,589,991.77 is obtained, representing the cumulative net profit over the 24-month period under optimal control. As a benchmark, under a static allocation scenario $(u = v = 0.5)$ maintained over the same period, $J_{static} =$ IDR 47183752716413.66 is obtained, so that the dynamic optimal allocation strategy yields a profit 1.1732% higher than this static baseline.

Table 4. Proportion of palm sap allocation at t-th month over a 24-month period

t	u	v	t	u	v	t	u	v	t	u	v
0	0.094	0.906	7	0.237	0.763	14	0.294	0.706	21	0.241	0.759

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t	u	v	t	u	v	t	u	v	t	u	v
1	0.119	0.881	8	0.251	0.749	15	0.294	0.706	22	0.231	0.769
2	0.142	0.858	9	0.263	0.737	16	0.291	0.709	23	0.333	0.667
3	0.164	0.836	10	0.273	0.727	17	0.286	0.714	24	0.5	0.5
4	0.185	0.815	11	0.282	0.718	18	0.278	0.722			
5	0.204	0.796	12	0.288	0.712	19	0.268	0.732			
6	0.221	0.779	13	0.292	0.708	20	0.256	0.745			

Figure 3 shows the dynamics of the quantities of palm sap, palm sugar, and bioethanol over 24-month period. The amount of palm sap remains constant at 170,291,000 liters, since $\eta = \theta$ makes the rate of change of N independent of the split between u and v, and the initial value N_0 lies exactly at its equilibrium point. The quantities of palm sugar and bioethanol essentially track the allocation proportions $u(t)$ and $v(t)$. palm sugar increases from its initial value to a peak of $3.85 \cdot 10^7$ kg at months 14–15, declines to $3.04 \cdot 10^7$ kg at month 22, then rises sharply to $8.16 \cdot 10^7$ kg at month 24. Conversely, bioethanol is relatively high at the outset, gradually declines to $7.22 \cdot 10^7$ liters at months 14–15, rises again to $7.84 \cdot 10^7$ liters at month 22, then decreases sharply to $4.06 \cdot 10^7$ liters at month 24.

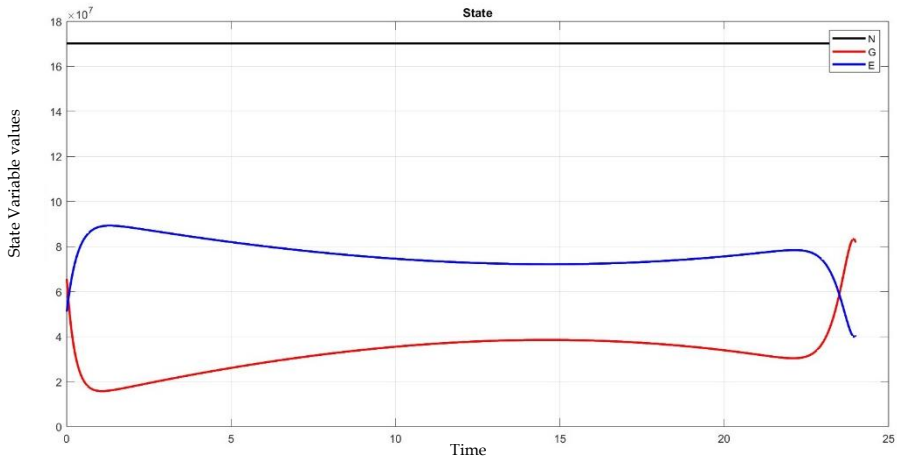


Figure 3. Dynamics of the quantities of palm sap, palm sugar, and bioethanol

4. Conclusions

This study aimed to determine an optimal dynamic allocation strategy for palm sap between palm sugar and bioethanol production that maximizes profit without sacrificing either commodity. The developed mathematical model was shown to possess a unique, positive, and bounded solution, as well as an asymptotically stable equilibrium point, indicating that the model provides a structurally consistent and economically meaningful representation of palm sap allocation dynamics. Using Pontryagin's Maximum Principle, the allocation of palm sap for bioethanol remained consistently higher than for palm sugar throughout the simulation period, consistent with the priority of renewable energy needs, while the allocation proportions for both commodities exhibited a non-monotonic, mutually inverse pattern (since $u(t) + v(t) = 1$), driven by the dynamics of the costate variables following the time-varying price forecasts, converging to a balanced 50%-50% allocation at the end of the horizon as a consequence of the transversality condition.

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This optimal allocation strategy yielded a profit approximately 1.17% higher than under a static 50%-50% allocation scenario maintained over the same period. These results demonstrate that the developed optimal control framework can serve as a quantitative decision-support tool for stakeholders in designing palm sap allocation policies that balance renewable energy needs with the sustainability of the palm sugar industry.

Several limitations of this study include the estimation of certain model parameters, particularly the palm sap allocation cost coefficient, based on assumptions due to the limited availability of actual operational cost data, as well as the moderate predictive accuracy of the bioethanol price model ($R^2 = 0.6329$), furthermore, the model is deterministic and does not yet account for future price uncertainty. Future research is recommended to extend this model by incorporating stochastic price disturbances or comparing the PMP strategy with a model predictive control approach to evaluate the robustness of the allocation strategy under market uncertainty.

Author Contributions

All authors contributed meaningfully to the conception, execution, and reporting of this study. Specific roles are described below using the CRediT taxonomy. Dia Ayu Nazihah: Conceptualization, Methodology, Data curation, Formal analysis, Visualization, Writing - original draft, Writing – editing. Aisyah Fikriyah Zahirah: Writing – original draft, Writing - editing. Ni Luh De Siska Sari Dewi: Writing – original draft, Writing- editing. Mardlijah: Promotor, Methodology, Supervision, Validation, Writing - review.

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Declaration of Competing Interest

The authors confirm that there are no competing financial interests or personal relationships that may have influenced the findings reported in this paper.

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