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Domination Number of Harary Graph

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ABSTRACT

The domination number of a graph G is the minimum number of vertices from any minimal dominating set in G. A subset of the vertex set S of G is referred to as a dominating set in G if any element in S dominates all vertices of G, implying that any vertex of G that does not contain an element of S is related and one distance from S. The domination number has become interesting research studies on several graphs k-connected such as circulant graphs, grids, and wheels. This study aims to identify the domination number of each k-connected graph represents Harary graph. The method used pattern detection and axiomatic deduction. The obtained results are two lemmas for a theorem in the domination theory of Harary graph. The discussion obtained challenges new patterns of the smallest of domination number of Harary Graph, especially shape $H_{2,n}$, $H_{4,n}$, and $H_{k,n}$ for every n vertices, k even integer, and $k < n$.

1. Introduction

Graph theory is a field of mathematics which studies two geometry objects, that is, vertices (the one is vertex) and edges. The graph notation, written G and interpreted graph G, is a pair (in fact, a sequenced combination) of two sets are the vertex set V and the edge set E of a specific graph G or could be represented by $G = (V, E)$. The vertex set V of G or $V(G)$ refers to the finite not-empty set V of objects named vertices of G. The edge set E of G or $E(G)$ represents a set (perhaps vacant) of an unordered pair of distinct vertices in $V(G)$. The number of vertices in G is commonly referred to as the order of G and is expressed by $p(G)$, but the number of edges represents its size and is expressed by $q(G)$.

The edge uv (or vu), defined $e = (u, v)$, connects a vertex u along with a vertex

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v termed adjacent. The vertices between u and v are dubbed as neighbors within each other. In this instance, the edge $e = (u, v)$ attaches the vertex u along with the edge e (as well as v and e) are considered to be incident with each other. The degree of a vertex v in G is the number of edges incident to v and is expressed by $\deg_G v$ or merely by $\deg(v)$ if the graph G is obvious from the context. The minimum degree of G, symbolized $\delta(G)$, is the low degree among the vertices of G. If each vertex in G has the same degree, then G is a regular graph. But, if it has the distinct degree then G is an irregular graph. The distance from u to v is the smallest length of each $u - v$ path in G and can be expressed by $d_G(u, v)$ or merely $d(u, v)$ if the graph G under concern is obvious. The number of vertices in G, the vertices both u and v are adjacent if the edge uv exists in $E(G)$ and is termed the cardinality of V. At times, it is beneficial to compose $|V|$ (Snyder, 2011).

One of the interesting topics in the development concerning graph theory followed by its implications involves how the domination number of a graph could be applied to on-line social networks (OSNs) like Facebook to determine asymptotic sublinear bounds (Bonato et al., 2015). The domination number of a graph is employed as well in digraphs (Hao, 2017), tournaments (Chudnovsky et al., 2018), and games (Alon et al., 2002) (Xu & Li, 2018) (Haynes et al., 2021). The original domination number appears in the 1850s, when chess enthusiasts in Europe studied the Five Queens Issue is to discover a minimal domination set for five queens on the standard 8×8 squares chessboard (Haynes et al., 1998). According to graph theory, chess piece (queens) represents vertices and paths of displacement between squares on a chessboard represent edges. The minimum number of chess piece (queens) of mutually colliding with each other in a single move represents the domination number of a dominating set in G. For further details, see (Chartrand et al., 2019) (Haynes et al., 2023).

The domination number of several graphs which have been explored include: circulant graphs (Rad, 2009), grid graphs (Gonçalves et al., 2011) (Snyder, 2011), and wheel graphs (Kalyan & James, 2019). The graphs are included in k-vertex-connected graph or are sometimes called k-connected graph. With regard to a nonnegative integer k, a graph G is considered to be k-connected if $\kappa(G) \geq k$. (The symbol κ becomes the Greek letter kappa). If G is a connected graph, the connectivity $\kappa(G)$ in G can be defined as the size of a minimum vertex-cut of G. If $G = K_n$ (complete graph of order n) with any positive integer n, then $\kappa(G)$ is deemed to be $n - 1$. Accordingly, for any graph G within order n, $0 \leq \kappa(G) \leq n - 1$ (Chartrand & Zhang, 2012).

Another example of k-connected graph gets a minimum degree k and order n for integer k and n using $2 \leq k < n$, but can have the bound upon the domination number is less than the general bound that is Harary graph. This research aims upon the domination number of the construction of known as Harary graph $H_{k,n}$ within the specified case (k even) and n vertices for integer $k \geq 0$ and $k < n$.

Dominating Set and Domination Number of a Graph

Suppose that the vertex set S in G, $S \subseteq V(G)$, becomes a dominating set if each vertex of G is dominated by certain vertex in S. Likewise, a set S of vertices in G is a dominating set of G if each vertex in $V(G) - S$ is adjacent with certain vertex in S (Chartrand & Zhang, 2012). A vertex in a set S is considered to dominate its own and each of its neighbors. Thus, if a set S becomes a dominating set in G, then each vertex of G is dominated by at least one vertex of S (Henning & Vuuren, 2022). For

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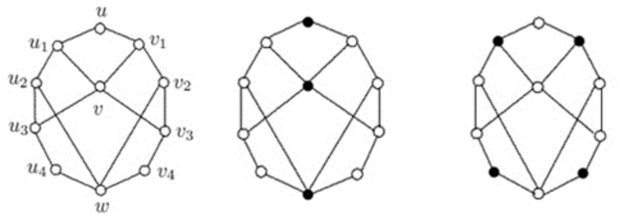
instance, according to the graph G on Figure 1(a). The set $S_1 = \{u, v, w\}$ in Figure 1(b) and the set $S_2 = \{u_1, u_4, v_1, v_4\}$ in Figure 1(c), denoted by solid vertices, are both dominating set in G (Chartrand & Zhang, 2012).



(a) **(b)** **(c)**

Figure 1. Two dominating sets in a graph G

The domination number of G, is expressed by $\gamma(G)$, which means the cardinality of a minimal dominating set in a graph G. Since a set S of vertices in G acts as a dominating set, the domination number is established for every graph. If G is a graph of order n, subsequently $1 \leq \gamma(G) \leq n$. A graph G of order n gets domination number 1 if and only if G includes a vertex v of degree $n - 1$, in which case $\{v\}$ is a minimal dominating set; whereas $\gamma(G) = n$ if and only if a graph G is not a complete graph, written $G = \overline{K_n}$, in which context $V(G)$ is the unique (minimum) dominating set (Chartrand & Zhang, 2012).

In Figure 1(b), the set $S_1 = \{u, v, w\}$ becomes a dominating set for G. Thereby, $\gamma(G) \leq 3$ because it has no dominating set in two vertices. To show that the order of G has 11 and that the degree of each vertex of G lies at most 4. This implies that no vertex may dominate any more than 5 vertices and that each two vertices dominate aa much as possible 10 vertices. That is, $\gamma(G) > 2$ and so $\gamma(G) = 3$ (Chartrand & Zhang, 2012).

Connected Graphs and Connectivity

Each graph G includes certain subgraph of G. A graph H is defined as the subgraph of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A maximal connected subgraph of G is termed a component in G. Two vertices u and v in G are linked if $u = v$, or if $u \neq v$ and a u-v path exist in G. The number of components in G is symbolized by $k(G)$.

A subgraph H becomes a component in G if H is not included within other subgraphs to have vertices and edges more than H. A graph G is termed the connected graph G if G includes only one component. A graph G can be a disconnected graph because a vertex-cut in G. A vertex-cut in G is a set S of vertices, where $S \subseteq V(G)$, so that $G - S$ includes more than one component.

The connectivity in G, expressed $\kappa(G)$, is assosiated to two terminologies are vertex-connectivity and edge-connectivity in a connected graph G. The smallest number of vertices, whose elimination can either disconnect G or reduce it to a 1-vertex graph is termed the vertex-connectivity in a connected graph G, symbolized $\kappa_v(G)$. While the edge-connectivity in a connected graph G, symbolized $\kappa_e(G)$, is the minimal number of edges whose elimination can disconnect G. If a graph G is linked and includes $\kappa(G) \geq k$, then a graph G is known as k-connected graph G. One example for a k-connected graph G is Harary graph.

Theorem 1. (Harary, 1962) (West, 2001) If $|G| = n$ and $\kappa(G) = k$, then size of G,

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$\|G\| \geq \left\lceil \frac{nk}{2} \right\rceil$.

Proof:
Based on the concept of connectivity graph that $\kappa(G) \geq \delta(G)$ obtained $\delta(G) \geq k$ then

$$2\|G\| = \sum_{v \in V(G)} deg(v) \geq \sum_{v \in V(G)} \delta(G) = |G|\delta(G) \geq nk$$

Therefore, $\|G\| \geq \frac{nk}{2}$. Because $\|G\|$ is a positive integer, then $\left\lceil \frac{nk}{2} \right\rceil$ edge ∎

Theorem 1 gives lower bound for size of G in term of the ceiling function, symbolized $\left\lceil \frac{nk}{2} \right\rceil$, that means the minimum integer more than or equal to $\frac{nk}{2}$. The ceiling function of a real number x, symbolized $\lceil x \rceil$, means the unique integer n such that $n - 1 < x \leq n$. While the floor function of a real number x, symbolized $\lfloor x \rfloor$, means the unique integer n such that $n \leq x < n + 1$. Intuitively, the ceiling function as rounding up and the floor function as rounding down (Saoub, 2021).

Theorem 1 proves that Harary graph achieves the smallest possible number of edges. For completeness, this will present the proofs for all cases, that is, the case when k is even and the another case when k is odd.

Proof case 1: k is even, $k = 2r$. Let $= H_{n,k}$. since $\delta(G) = k$, it suffices to prove $\kappa(G) \geq k$. For $S \subseteq V(G)$ with $|S| < k$, this will prove that $G - S$ is connected. Consider $u, v \in V(G) - S$. The original circular arrangement has a clockwise u, v-path and a counterclockwise u, v-path along the circle. Let A and B be the sets of internal vertices on these two paths.

Since $|S| < k$, the pigeonhole principle implies that in one of $\{A, B\}$, S has fewer than $\frac{k}{2}$ vertices. Since in G each vertex has edges to the next $\frac{k}{2}$ vertices in a particular direction, deleting fewer than $\frac{k}{2}$ consecutive vertices cannot block travel in that direction. Thus, this can find a u, v-path in $G - S$ via the set A or B in which S has fewer than $\frac{k}{2}$ vertices.

Proof case 2: k is odd, for $n > k = 2r + 1$ and $r \geq 1$. This will show that the Harary graph $H_{k,n}$ is k-connected. The graph consists of n vertices $v_0, \dots, v_{n-1}$ spaced equally around a circle, with each vertex adjacent to the r nearest vertices in each direction, plus the special edges $v_i v_{i+\left\lfloor \frac{n}{2} \right\rfloor}$ for $0 \leq i \leq \left\lfloor \frac{(n-1)}{2} \right\rfloor$. When n is odd, $v_{\left\lfloor \frac{n}{2} \right\rfloor}$ has two incident special edges.

To prove that $\kappa(G) = k$, consider a separating set S. Since $G - S$ is disconnected, there are nonadjacent vertices x and y such that every x, y-path passes through S. Let $C(u, v)$ denote the vertice encountered when moving from u to v clockwise along the circle (except u and v). The cut S must contain r consecutive vertices from each of $C(x, y)$ and $C(y, x)$ in order to break every x, y-path (otherwise, one could start at x and always take a step in the direction of y). Hence $|S| \geq k$ unless S contains exactly r consecutive vertices in each of $C(x, y)$ and $C(y, x)$.

In this case, this claims that there remains an x, y-path using a special edge involving x or y. Let x' and y' be the neighbors of x and y along the special edges, using v_0 as the neighbor when one of these is $v_{\left\lfloor \frac{n}{2} \right\rfloor}$. Label x and y so that $C(x, y)$ is smaller than $C(y, x)$ (diametrically opposite vertices require n even and are

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adjacent). Note that $|C(x',y')| \geq |C(x,y)| - 1$ (the two sets have different sizes when n is odd if $x = v_i$ and $y = v_j$ with $0 \leq j < \left\lceil \frac{n}{2} \right\rceil \leq i \leq n-1$). Because $|C(x,y)| \geq r$, this has $|C(x',y')| \geq r - 1$. Therefore, when this deletes r consecutive vertices from $C(y,x)$, all of $C(y,x') \cup \{x'\}$ or $\{y'\} \cup C(y',x)$ remains. Therefore at least one of the two x,y-path with these sets as the interval vertices remains $G - S$ ■

Harary Graph

Frank Harary provided the steps for forming a k-connected graph that is labeled $H_{k,n}$ on n vertices that has precisely $\left\lceil \frac{nk}{2} \right\rceil$ edges for $k \geq 2$ (Harary, 1962). The form of the Harary graph $H_{k,n}$ initiates by creating an n-cycle graph, where the vertices are respectively numbered $0,1,2,\dots,n-1$ clockwise within its perimeter. The form of $H_{k,n}$ relies on the equality of k and n (Gross et al., 2019).

The Harary graph $H_{k,n}$ represents a k-connected graph to $\kappa(G) = k$ and provided $k < n$, point n vertices alongside a circle, alike spaced. The form of a Harary graph $H_{k,n}$ consists three cases (West, 2001):

i. If k even, construct $H_{k,n}$ by making every vertex adjacent to the closest $\frac{k}{n}$ vertices in every direction within the circle.
ii. If k is odd but n is even, construct $H_{k,n}$ by making every vertex adjacent to the closest $\frac{(k-1)}{2}$ vertices in every direction along with to the diametrically contrary vertex.
iii. If k and n become both odd, index the vertices with the integers modulo n. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ within $0 \leq i \leq (n-1)/2$.

For instance, the form of Harary graphs $H_{k,n}$ for three distinct types of cases are the graphs $H_{4,8}$, $H_{5,8}$, and $H_{5,9}$ as outlined by (Tanna, 2015).

2. Method

This study uses pattern detection method through searching for a dominating set then determining the minimum cardinality. Another method applied in this study is the use of axiomatic deductive method based on the principles of deductive proof. The result will be obtained in the form of two lemmas and a new theorem that have been proven deductively so that its truth generally applies. The basis for these results uses theorem 1, concepts, and theorems to construct Harary graph $H_{k,n}$. These can be seen for further details in (West, 2001)(Tanna, 2015).

The study design consists of (i) initialize the construction of Harary graphs $H_{k,n}$ which was chosen; (ii) find and analyze a dominating set of Harary graph $H_{k,n}$ using pattern detection and then axiomatic deductive proofs; (iii) find the vertex with maximum degree as the dominating vertex in Harary graph $H_{k,n}$; (iv) find the next maximum degree vertex that has not been dominated; (v) determine the dominating set in the Harary graph $H_{k,n}$; (vi) determine the order number or the cardinality of Harary graph $H_{k,n}$; (vii) get the domination number of Harary graph $H_{k,n}$; (viii) make the conclusion. More generally, this study design could be simplified in Figure 2.

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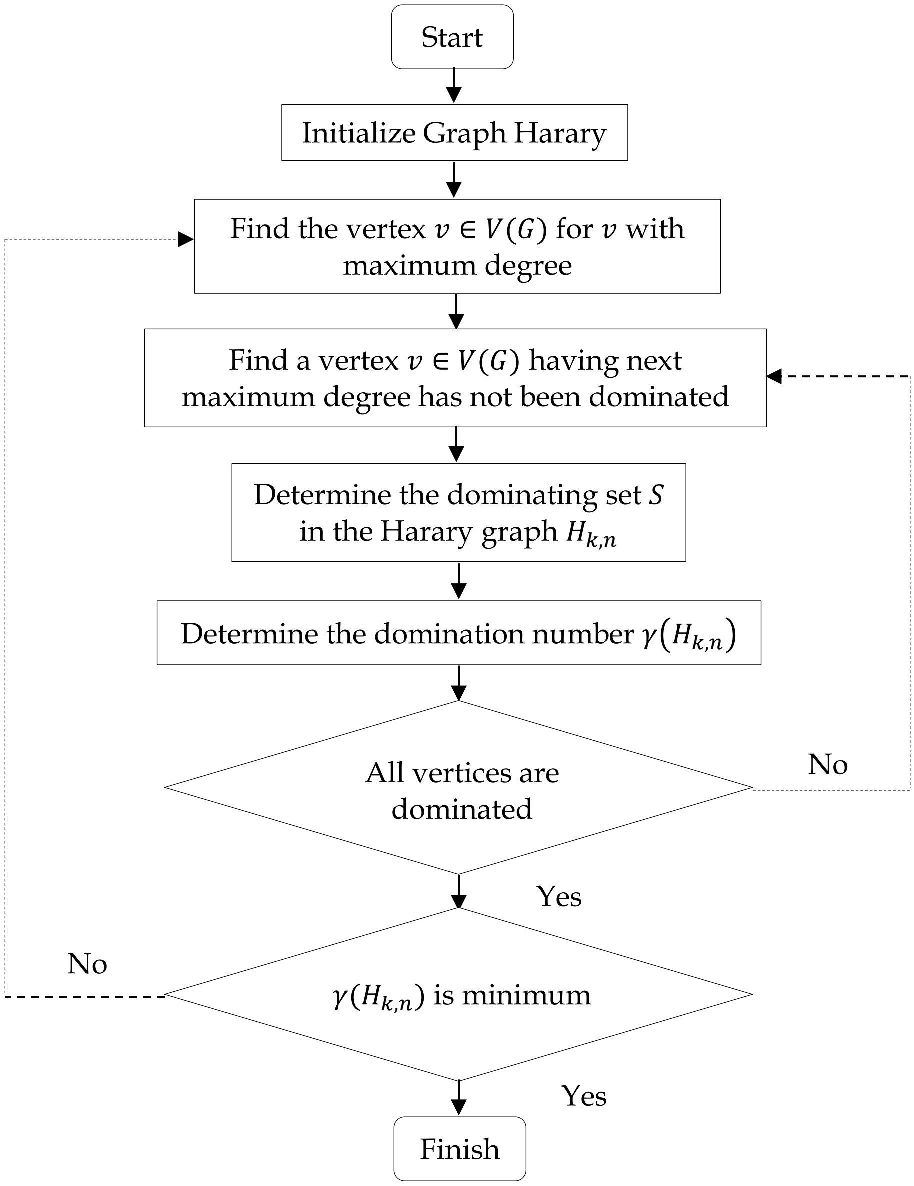

Figure 2. The Study Design

3. Results and Discussion

This section contain the constructions of Harary graphs $H_{k,n}$ for k even and $k < n$ that are explained in the following two lemmas and a new theorem along its proofs. The first lemma in the proof is strengthened by the results as in (Chartrand & Zhang, 2012), because the Harary graph $H_{k,n}$ in lemma 1 implies that it is 2-regular.

The second lemma was inspired from an old result for domination numbers by (Rad, 2009). While theorem 2 is proven as in (Khodkar et al., 2007) (Huang & Xu, 2008) that the Harary graph $H_{k,n}$ in theorem 2 implies that it is k-regular (Tanna, 2015).

Lemma 1. Let the Harary graph $H_{k,n}$ with $\kappa(H_{k,n}) = 2$ and n vertices, then the domination number of Harary graph $H_{2,n}$, is expressed by $\gamma(H_{2,n})$, is $\left\lceil \frac{n}{3} \right\rceil$.

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Proof:
Suppose that the Harary graph for $k < n$ and k even such that the order of Harary graph $H_{2,n} \geq 3$, for $k = 2$. Case $n = 3$, then the Harary graph becomes $H_{2,3}$. Since the construction of Harary graph $H_{k,n}$ for k even, every vertex to its adjacent $\frac{k}{2}$ neighbors in every direction around the circle. That is, if an arbitrary vertex $v_i \in V(H_{2,3})$ is taken, then v_i surely dominates the vertices $v_{i\pm1}, v_i$. If the set $S \subseteq V(H_{2,3})$ and $S = \{v_{i\pm1}, v_i\}$, then $|S| = 3$. Therefore, the domination number $\gamma(H_{2,3}) = 1$.

For the Harary graph $H_{2,4}$, there is an vertex v_j dominates itself and a set $S_1 \subseteq S$ dominates the vertices $v_{i\pm1}, v_i$ and the set $S \subseteq V(H_{2,4})$. So that the set $S = S_1 \cup v_j$ dominates $H_{2,4}$. Therefore, the domination number $\gamma(H_{2,4}) = 2$.

For the Harary graph $H_{2,5}$, there is an vertex v_j dominates the vertices v_{j-1}, v_j and a set S_1 dominates the vertices in $H_{2,5}$. For the Harary graph $H_{2,6}$, there is an vertex v_j or a set $S_2 \subseteq V(H_{2,6})$ dominates the vertices $v_{j\pm1}, v_j$ and a set S_1 dominates the vertices in $H_{2,6}$. Then the $S = S_1 \cup v_j$ dominates $H_{2,5}$ and the set $S = S_1 \cup S_2$ dominates $H_{2,6}$. Therefore, the domination numbers of the graphs $H_{2,5}$ and $H_{2,6}$ are $\gamma(H_{2,5}) = \gamma(H_{2,6}) = 2$.

For the Harary graph $H_{2,7}$, there is an vertex w in $H_{2,7}$ that dominates itself and the set $S = S_1 \cup S_2$ dominates $H_{2,7}$. Therefore, the domination number $\gamma(H_{2,7}) = 3$.

The sequence of $\gamma(H_{2,n})$ for $n = 3,4,5,6,7$ was obtained 1,2,2,2,3. By continuing to repeat iterations like the previous steps for $n > 7$, the sequence pattern of the domination number in $H_{2,n}$ is $\left\lceil \frac{n}{3} \right\rceil$. ■

Lemma 2. Let the Harary graph $H_{k,n}$ with $\kappa(H_{k,n}) = 4$ and n vertices, then the domination number of Harary graph $H_{4,n}$, is expressed by $\gamma(H_{4,n})$, is $\left\lceil \frac{n}{5} \right\rceil$.

Proof:
For $k = 4$, the order of Harary graph $H_{4,n} \geq 5$. Case $n = 5$, then the Harary graph becomes $H_{4,5}$. Since the construction of Harary graph $H_{k,n}$ for k even, every vertex to its adjacent $\frac{k}{2}$ neighbors in every direction around the circle. That is, if an arbitrary vertex $v_i \in V(H_{4,5})$ is taken, then v_i surely dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$. If the set $S \subseteq V(H_{4,5})$ and $S = \{v_{i\pm1}, v_{i\pm2}, v_i\}$, then $|S| = 3$. Therefore, the domination number $\gamma(H_{4,5}) = 1$.

For the Harary graph $H_{4,6}$, there is a set $S_1 \subseteq S$ that dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$ where the set $S \subseteq V(H_{4,6})$ and a vertex v_j dominates itself. That is, the set $S = S_1 \cup v_j$ dominates $H_{4,6}$ with the domination number $\gamma(H_{4,6}) = 2$.

For the Harary graph $H_{4,7}$, there is a set $S_1 \subseteq S$ that dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$ where the set $S \subseteq V(H_{4,7})$ and a vertex v_j dominates an vertex v_{j-1} and itself. That is, the set $S = S_1 \cup v_j$ dominates $H_{4,7}$ with the domination number $\gamma(H_{4,7}) = 2$.

For the Harary graph $H_{4,8}$, there is a set $S_1 \subseteq S$ that dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$ where the set $S \subseteq V(H_{4,8})$ and a vertex v_j dominates the vertices

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$v_{j\pm1}$ and itself. That is, the set $S = S_1 \cup v_j$ dominates $H_{4,8}$ with the domination number $\gamma(H_{4,8}) = 2$.

For the Harary graph $H_{4,9}$, there is a set $S_1 \subseteq S$ that dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$ where the set $S \subseteq V(H_{4,9})$ and a vertex v_j dominates the vertices $v_{j\pm1}, v_{j-2}$ and itself. That is, the set $S = S_1 \cup v_j$ dominates $H_{4,9}$ with the domination number $\gamma(H_{4,9}) = 2$.

For the Harary graph $H_{4,10}$, there is a set $S_1 \subseteq S$ that dominates the vertices $v_{i\pm1}, v_{i\pm2}, v_i$ where the set $S \subseteq V(H_{4,10})$ and a vertex v_j or a set $S_2 \subseteq V(H_{4,10})$ dominates the vertices $v_{j\pm1}, v_{j\pm2}, v_j$. That is, the set $S = S_1 \cup S_2$ dominates $H_{4,10}$ with the domination number $\gamma(H_{4,10}) = 2$.

For the Harary graph $H_{4,11}$, there is the set $S = S_1 \cup S_2$ that dominates $H_{4,11}$ and an vertex w in $H_{4,11}$ dominates itself. That is, the set $S = S_1 \cup S_2 \cup w$ dominates $H_{4,11}$ with the domination number $\gamma(H_{4,11}) = 3$.

The sequence of $\gamma(H_{4,n})$ for $n = 5,6,7,8,9,10,11$ was obtained 1,2,2,2,2,2,3. By continuing to repeat iterations like the previous steps for $n > 11$, the sequence pattern of the domination number in $H_{4,n}$ is $\left\lceil \frac{n}{5} \right\rceil$. ∎

Theorem 2. Let the Harary graph $H_{k,n}$ with $\kappa(H_{k,n}) = k$ for k even and n vertices, the the domination number of Harary graph $H_{k,n}$, is expressed by $\gamma(H_{k,n})$, is $\left\lceil \frac{n}{k+1} \right\rceil$.

Proof:
It will be shown that the domination number in $H_{k,n}$ is $\left\lceil \frac{n}{k+1} \right\rceil$ for k even which is connectivity each vertex and n specifies the number of vertices from the form of Harary graph, so $(k + 1)$ is odd. By means of induction, consider that $k = 2r$ and $r \in \mathbb{N}$ (natural set).

i. Consider that the argument is correct for $r = 1$, then
$\left\lceil \frac{n}{k+1} \right\rceil = \left\lceil \frac{n}{2+1} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$, 3 odd
ii. Consider that the argument is correct for an arbitrary $r = a$ and $a \in \mathbb{N}$, then
$\left\lceil \frac{n}{k+1} \right\rceil = \left\lceil \frac{n}{2r+1} \right\rceil = \left\lceil \frac{n}{2a+1} \right\rceil$, $2a + 1$ odd
iii. Consider that the argument is correct for an arbitrary $r = a + 1$ and $a \in \mathbb{N}$, then
$\left\lceil \frac{n}{k+1} \right\rceil = \left\lceil \frac{n}{2r+1} \right\rceil = \left\lceil \frac{n}{2(a+1)+1} \right\rceil = \left\lceil \frac{n}{2a+2+1} \right\rceil = \left\lceil \frac{n}{2a+3} \right\rceil$, $2a + 3$ odd

Based on the arguments i, ii, and iii, then it is confirmed that the domination number in Harary graph $H_{k,n}$ is $\left\lceil \frac{n}{k+1} \right\rceil$ for k even and n specifies the number of vertices of Harary graph $H_{k,n}$ with $(k + 1)$ odd. ∎

4. Conclusions

Fist case of Harary graph $H_{k,n}$ with constructing k even and provided $k < n$ for n vertices, gives two lemmas and a new theorem. The first lemma showed that the domination number of Harary graph $H_{2,n}$ for $\kappa(H_{2,n}) = 2$, which is expressed by $\gamma(H_{2,n}) = \left\lceil \frac{n}{3} \right\rceil$. The second lemma is obtained that the domination number of Harary graph $H_{4,n}$ for $\kappa(H_{4,n}) = 4$, which is symbolized by $\gamma(H_{4,n}) = \left\lceil \frac{n}{5} \right\rceil$.

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While a new theorem in the domination theory of Harary graph describes that each Harary graph $H_{k,n}$ consists the case 1 has the domination number, written $\gamma(H_{k,n})$, is $\left\lceil\frac{n}{k+1}\right\rceil$. This result is the consequences of two lemmas, that is, lemma 1 and lemma 2.

The next study recommends to construct and to prove the domination number in Harary graph $H_{k,n}$ in which each value k is odd but each value n is even. Another construction which can also be proven the domination number of $H_{k,n}$ if k and n were odd. Those cases may use previous results to prove the domination numbers as in (Khodkar et al., 2007).

Author Contributions

The first author conducted an understanding of the concepts and studying literatures, observed about this study, and wrote article manuscripts. While the second author helped studying literatures, evaluating the analysis that has been done in the observation step, and disseminating the results of this study.

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Declaration of Competing Interest

The authors declare that there are no conflicts of interest.

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