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**Domination Number of Harary Graph**

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| **ARTICLE INFO** (11 pt) |  | **ABSTRACT** (11 pt) |
| |  |  |  | | --- | --- | --- | | **Article History** | | | | Received | : |  | | Revised | : |  | | Accepted | : |  | | Available Online | : |  | |  | The domination number of graph is the smallest cardinality of a dominating set of graph . A subset of a vertex set of is called a dominating set if every element of dominates every vertex of , meaning that every vertex of that is not an element of is connected and one distance from . The domination number has become interesting research studies on several graphs -connected such as circulant graphs, grids, and wheels. This study aims to determine domination number of the other -connected graph is Harary graph. The method used pattern detection and axiomatic deduction. The obtained results are new lemmas and theorems. The discussion obtained challenges new patterns of the smallest of domination number of Harary Graph, especially shape , , and for every vertices, even integer, and . |
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1. **Introduction**

Graph theory is a branch of mathematics that discusses two geometry objects, namely, vertices (the singular is vertex) and egdes. The graph notation, written and read graph , is a pair (actually an ordered pair) of two sets are the vertex set and edge set of a particular graph or could be denoted by . The vertex set of or is a finite nonempty set of objects called vertices of . The edge set of or is a set (maybe empty) of an unordered pair of different vertices in . The number of vertices in is often called order of and is denoted by , while the number of edges is its size and is denoted by .

An edge (or ), denoted , joins a vertex and a vertex called adjacent. The vertices and are referred to as neighbors of each other. In this case, the edge attaches the vertex and the edge (as well as and ) are said to be incident with each other. The degree of an vertex in is the number of edges incident with and is denoted by or simply by if the graph is clear from the context. The minimum degree of , denoted , is the minimum degree among the vertices of . The distance between and is the smallest length of any path in and is denoted by or simply if the graph under consideration is clear. The number of vertices in , the vertices and are adjacent if the edge is in and is called the cardinality of . At times, it is useful to write (Snyder, 2011).

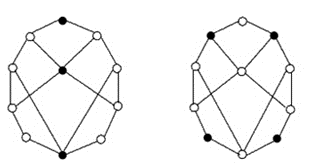
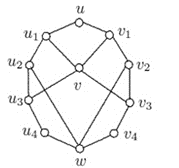
One of the interesting topics in the development of graph theory and its applications is the domination number of a graph could be applied to on-line social networks (OSNs) like Facebook to determine asymptotic sublinear bounds (Bonato et al., 2015). The domination number of a graph is also applied in digraphs (Hao, 2017), tournaments (Chudnovsky et al., 2018), and games (Alon et al., 2002) (Xu & Li, 2018) (Haynes et al., 2021). The original domination number appears in the 1850s, when chess enthusiasts in Europe studied the Five Queens Problem is to find a minimum domination set of five queens on a standard 8 8 squares chessboard (Haynes et al., 1998). According to graph theory, chess piece (queens) represents vertices and paths of displacement between squares on a chessboard represent edges. The minimum number of chess piece (queens) of mutually colliding with each other in a single move is the domination number of a dominating set of . For further details, see (Chartrand et al., 2019) (Haynes et al., 2023).

The domination number of several graphs which have been studied include: circulant graphs (Rad, 2009), grid graphs (Gonçalves et al., 2011) (Snyder, 2011), and wheel graphs (Kalyan & James, 2019). The graphs are included in -vertex-connected graph or are sometimes called -connected graph. For a nonnegative integer , a graph is said to be -connected if . (The symbol is the Greek letter kappa). If is a connected graph, the connectivity of is defined as the size of a smallest vertex-cut of . If (complete graph of order ) for some positive integer , then is defined to be . Therefore, for every graph of order , (Chartrand & Zhang, 2012).

Another example of -connected graph has a minimum degree and order for integer and with , but can have the bound on the domination number is less than the general bound that is Harary graph. This research aims on the domination number of the construction of the Harary graph within the given case ( even) and vertices for integer and .

**Dominating Set and Domination Number of a Graph**

Suppose that the vertex set of , , is a dominating set if every vertex of is dominated by some vertex in . Equivalently, a set of vertices of is a dominating set of if every vertex in is adjacent to some vertex in (Chartrand & Zhang, 2012). A vertex in a set is said to dominate itself and each of its neighbours. Thus, if a set is a dominating set in , then every vertex of is dominated by at least one vertex of (Henning & Vuuren, 2022). For example, in the graph of Figure 1(a). the set in Figure 1(b) and the set in Figure 1(c), indicated by solid vertices, are both dominating set in (Chartrand & Zhang, 2012).



|  |  |  |
| --- | --- | --- |
| **(a)** | **(b)** | **(c)** |

**Figure 1.** Two dominating sets in a graph

The domination number of , is denoted by , is the cardinality of a minimum dominating set in a graph . Since a set of vertices of is always a dominating set, the domination number is defined for every graph. If is a graph of order , then . A graph of order has domination number 1 if and only if contains a vertex of degree , in which case is a minimum dominating set; while if and only if a graph is not a complete graph, denoted , in which case is the unique (minimum) dominating set (Chartrand & Zhang, 2012).

In Figure 1(b), the set is a dominating set for . Therefore, because there is no dominating set with two vertices. To show that the order of is 11 and that the degree of every vertex of is at most . This means that no vertex can dominate more than 5 vertices and that every two vertices dominate at most 10 vertices. That is, and so (Chartrand & Zhang, 2012).

**Connected Graphs and Connectivity**

Every graph contains some subgraph of . A graph is called a subgraph of a graph , written , if and . A maximal connected subgraph of is called a component of a graph . Two vertices and in a graph are connected if , or if and a - path exist in . The number of components of is denoted by .

A subgraph becomes a component of a graph if is not contained in other subgraphs to have vertices and edges more than . A graph is called a connected graph if has only one component. A graph can be a disconnected graph because a vertex-cut in a graph . A vertex-cut in a graph is a set of vertices, where , such that has more than one component.

The connectivity of a graph , written , is related to two terminologies are vertex-connectivity and edge-connectivity of a connected graph . The minimum number of vertices, whose removal can either disconnect or reduce it to a 1-vertex graph is called the vertex-connectivity of a connected graph , denoted . While the edge-connectivity of a connected graph , denoted , is the minimum number of edges whose removal can disconnect . If a graph is connected and has , then a graph is called a -connected graph . One example of a -connected graph is Harary graph.

**Theorem 1.** (Harary, 1962) (West, 2001) If and , then size of , .

Proof:

Based on the concept of connectivity graph that obtained then

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Therefore, . Because is a positive integer, then edge

Theorem 1 gives lower bound for size of in term of the ceiling function, denoted , that is the smallest integer greater than or equal to . The ceiling of a real number , denoted , is the unique integer such that . Intuitively, the ceiling function as rounding up (Saoub, 2021).

**Harary Graph**

Frank Harary gave a procedure for constructing a -connected graph that is denoted on vertices that has exactly edges for (Harary, 1962). The construction of the Harary graph begins with an -cycle graph, whose vertices are consecutively numbered clockwise around its perimeter. The construction of depends on the parity of and (Gross et al., 2019).

The Harary graph is a -connected graph with and given , place vertices around a circle, equally spaced. The construction of a Harary graph consists three cases (West, 2001):

1. If even, form by making each vertex adjacent to the nearest vertices in each direction around the circle.
2. If odd and is even, form by making each vertex adjacent to the nearest vertices in each direction and to the diametrically opposite vertex.
3. If and are both odd, index the vertices by the integers modulo . Construct from by adding the edges for .

For example, the constructions of Harary graphs for three different types of cases are the graphs , , and as described by (Tanna, 2015).

1. **Method**

This study uses pattern detection method through searching for a dominating set then determining the minimum cardinality. Another method applied in this study is the use of axiomatic deductive method based on the principles of deductive proof. The result will be obtained in the form of two lemmas and a new theorem that have been proven deductively so that its truth generally applies.

The study design consists of (i) initialize the construction of Harary graphs which was chosen; (ii) find and analyze a dominating set of the Harary graph using pattern detection and axiomatic deductive proofs; (iii) find the vertex with maximum degree as the dominating vertex in Harary graph ; (iv) find the next maximum degree vertex that has not been dominated; (v) determine the dominating set in the Harary graph ; (vi) determine the order number or the cardinality of Harary graph ; (vii) get the domination number of Harary graph ; (viii) make the conclusion. More generally, this study design could be simplified in Figure 2.

No

Yes

Start

Initialize Graph Harary

Find the vertex for with maximum degree

Find a vertex having next maximum degree has not been dominated

Determine the dominating set in the Harary graph

Determine the domination number

All vertices are dominated

is minimum

Yes

No

Finish

**Figure 2.** The Study Design

1. **Results and Discussion**

This section contain the constructions of Harary graphs for even and that are explained in the following two lemmas and a new theorem along its proofs.

**Lemma 1.** Let the Harary graph with and vertices, then the domination number of the Harary graph , is denoted by , is .

Proof:

Suppose that the Harary graph for and even such that the order of Harary graph , for . Case , then the Harary graph becomes . Since the construction of Harary graph for even, every vertex to its adjacent neighbors in every direction around the circle. That is, if an arbitrary vertex is taken, then surely dominates the vertices . If the set and , then . Therefore, the domination number .

For the Harary graph , there is an vertex dominates itself and a set dominates the vertices and the set . So that the set dominates . Therefore, the domination number .

For the Harary graph , there is an vertex dominates the vertices and a set dominates the vertices in . For the Harary graph , there is an vertex or a set dominates the vertices and a set dominates the vertices in . Then the dominates and the set dominates . Therefore, the domination numbers of the graphs and are .

For the Harary graph , there is an vertex in that dominates itself and the set dominates . Therefore, the domination number .

The sequence of ) for was obtained . By continuing to repeat iterations like the previous steps for , the sequence pattern of the domination number in is

**Lemma 2.** Let the Harary graph with and vertices, then the domination number of the Harary graph , is denoted by , is .

Proof:

For , the order of Harary graph . Case , then the Harary graph becomes . Since the construction of Harary graph for even, every vertex to its adjacent neighbors in every direction around the circle. That is, if an arbitrary vertex is taken, then surely dominates the vertices . If the set and , then . Therefore, the domination number .

For the Harary graph , there is a set that dominates the vertices where the set and a vertex dominates itself. That is, the set dominates with the domination number .

For the Harary graph , there is a set that dominates the vertices where the set and a vertex dominates an vertex and itself. That is, the set dominates with the domination number .

For the Harary graph , there is a set that dominates the vertices where the set and a vertex dominates the vertices and itself. That is, the set dominates with the domination number .

For the Harary graph , there is a set that dominates the vertices where the set and a vertex dominates the vertices and itself. That is, the set dominates with the domination number .

For the Harary graph , there is a set that dominates the vertices where the set and a vertex or a set dominates the vertices . That is, the set dominates with the domination number .

For the Harary graph , there is the set that dominates and an vertex in dominates itself. That is, the set dominates with the domination number .

The sequence of ) for was obtained . By continuing to repeat iterations like the previous steps for , the sequence pattern of the domination number in is

**Theorem 2.** Let the Harary graph with for even and vertices, the the domination number of the Harary graph , is denoted by , is .

Proof:

It will be shown that the domination number in is for even which is connectivity each vertex and is the number of vertices from the construction of Harary graph, so is odd. By way of induction, assume that and (natural set).

1. Assume that the assertion is true for , the

odd

1. Assume that the assertion is true for an arbitrary and , then

odd

1. Assume that the assertion is true for an arbitrary and , then

odd

Based on the assertions 1, 2, and 3, then it is proved that the domination number of Harary graph is for even and is the number of vertices of Harary graph with odd

1. **Conclusions**

The domination number of Harary graph, is denoted by , for even and vertices as follows:

1. The domination number of Harary graph is denoted by for .
2. The domination number of Harary graph is denoted by for .
3. The domination number of Harary graph is denoted by for .

**Author Contributions**

The first author conducted an understanding of the concepts and studying literatures, observed about this study, and wrote article manuscripts. While the second author helped studying literatures, evaluating the analysis that has been done in the observation step, and disseminating the results of this study.

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**Declaration of Competing Interest**

The authors declare that there is no conflict of interest in this study.

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